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On transient waves in dispersive media produced by moving point sources

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Abstract

A solution for the inhomogeneous telegraph equation for a point source moving with the velocity of light is constructed. We find relations describing both the transient and steady-state wave processes. The solutions obtained are used to define electromagnetic waves in a conductive medium. The case of a source moving faster than light is also given.

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1. Introduction

The aim of this paper is to construct transient solutions to the inhomogeneous telegraph and Maxwell's equations describing waves formed by a moving point source. We suppose that the source starts at some fixed moment of time and travels along a straight line with the velocity of light (for scalar waves, with the velocity of wavefront). Using the Olevsky theorem [1] we generalize the method applied for the construction of the analogous solutions in free space [2], we obtain an explicit solution of the initial-value problem to the 3D telegraph equation in space–time representation, and discuss the application of these solutions to the description of scalar and electromagnetic waves in dispersive media. A solution is also given for the source moving with a velocity greater than the velocity of light. The results obtained are similar to previously published solutions describing steady-state localized waves in dispersive media [3–5] and spherical waves in a conductive medium produced by sources on an expanding sphere [6].

2. Method used

First, we construct an axisymmetric solution of the initial-value problem in the cylindrical coordinates ρ, φ, z

$$\left(\partial_\tau^2 - (1/\rho)\partial_\rho(\rho\partial\rho) - \partial_z^2 - a^2\right)\psi = (4\pi/c)j \qquad \psi = j \equiv 0 \quad \tau < 0 \tag{1}$$

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where $\tau = ct$ is the time variable, c is the velocity of light and the real constant a^2 determines the wave dispersion. We write the source function j as

$$j = (1/(2\pi\rho))h(\tau)h(z)\delta(z - \beta\tau)f(\tau) \tag{2}$$

where $h(\tau)$ is the Heaviside function, $\delta(x)$ is the Dirac function, $f(\tau)$ is an arbitrary continuous function and $\beta > 0$ is an arbitrary parameter.

Separating the radial variable ρ by means of the Fourier–Bessel transform

$$g(\rho) = \int_0^\infty \mathrm{d}s\, s J_0(s\rho)g(s) \qquad g(s) = \int_0^\infty \mathrm{d}\rho\, \rho J_0(s\rho)g(\rho) \tag{3}$$

where $J_0(s\rho)$ is the Bessel function of the first kind, from (1) and (2) we obtain

$$\left(\partial_\tau^2 - \partial_z^2 + (s^2 - a^2)\right)\psi(s) = \frac{2}{c}h(z)h(\tau)\delta(z - \beta\tau)f(\tau) \qquad \psi(s) \equiv 0 \quad \tau < 0 \tag{4}$$

Using the Olevsky theorem (see the appendix) we write the Riemann function in the form

$$R = J_0(s\sqrt{(\tau - \tau')^2 - (z - z')^2}) - \int_{\tau-\tau'}^{z-z'} \mathrm{d}\xi \frac{\partial}{\partial\xi} I_0(a\sqrt{\xi^2 - (z - z')^2}) \cdot J_0(s\sqrt{(\tau - \tau')^2 - \xi^2})$$

where $I_0(x)$is the modified Bessel function, and we obtain the solution to problem (4) with the help of the Riemann formula

$$\psi(s) = \frac{1}{c}\int_0^\tau \mathrm{d}\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z' f(\tau')h(z')\delta(z' - \beta\tau')R(s, z, \tau, z', \tau').$$

Then the solution to problems (1) and (2) can be represented by the integral

$$\psi = \frac{1}{c}\int_0^\tau \mathrm{d}\tau' f(\tau') \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z' h(z')\delta(z' - \beta\tau')I(\rho, z', \tau') \tag{5}$$

where

$$I(\rho, z', \tau') = \int_0^\infty \mathrm{d}s\, s J_0(s\rho)J_0(s\sqrt{(\tau - \tau')^2 - (z - z')^2}) - \int_{\tau-\tau'}^{z-z'} \mathrm{d}\xi \frac{\partial}{\partial\xi} I_0(a\sqrt{\xi^2 - (z - z')^2}) \int_0^\infty \mathrm{d}s\, s J_0(s\rho)J_0(s\sqrt{(\tau - \tau') - \xi^2}).$$

Putting

$$\int_0^\infty \mathrm{d}s\, s J_0(s\rho)J_0(s\sqrt{(\tau - \tau') - x^2}) = \frac{1}{\rho}\delta(\rho - \sqrt{(\tau - \tau') - x^2}) \tag{6}$$

where x^2 denotes $(z - z')^2$ or ξ^2, from (5) we obtain

$$\psi = \frac{1}{c\rho}\int_0^\tau \mathrm{d}\tau' f(\tau') \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z' h(z')\delta(\beta\tau' - z') \times \Bigg[\delta(\rho - \sqrt{(\tau - \tau')^2 - (z - z')^2}) - \int_{\tau-\tau'}^{z-z'} \mathrm{d}\xi \frac{\partial}{\partial\xi} I_0(a\sqrt{\xi^2 - (z - z')^2})\, \delta(\rho - \sqrt{(\tau - \tau')^2 - \xi^2})\Bigg]. \tag{7}$$

This relation gives an algorithm for constructing a solution to the scalar problem (1) and (2). So, in the special case where a point source propagates with the velocity of light, $\beta = 1$, using (7) we obtain:

$$\psi = \psi_0 + \psi_a = h(\tau - r)\frac{1}{c(\tau - z)}f(\Phi) - \frac{1}{c\rho}\int_0^\tau \mathrm{d}\tau' f(\tau') \times \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z' h(z')\delta(z' - \tau') \times \int_{\tau-\tau'}^{z-z'} \mathrm{d}\xi\, \delta(\rho - \sqrt{(\tau - \tau')^2 - (z - z')^2}) \frac{\partial}{\partial\xi} I_0(a\sqrt{\xi^2 - (z - z')^2}) \tag{8}$$

where $\Phi = \frac{1}{2}\left(\tau + z - \frac{\rho^2}{\tau - z}\right)$, $r = \sqrt{\rho^2 + z^2}$. The term ψ_0 is the known solution of the inhomogeneous wave equation (see [2, 7]), which describes the family of localized waves of Brittingham's type in free space. We calculate the item ψ_a by using the property of the δ-function and find the limits of integration with respect to the variable τ' with the help of the (τ', z')-plane diagram (see [2] for details). Summarizing the results, we express the scalar solution ψ in the form

$$\psi = h(\tau - r)\frac{1}{c(\tau - z)} f(\Phi) + \frac{a}{c\sqrt{2(\tau - z)}} \int_0^{\Phi} \mathrm{d}\tau' f(\tau') \frac{1}{\sqrt{\Phi - \tau'}} I_1(a\sqrt{2(\tau - z)}\sqrt{\Phi - \tau'}). \tag{9}$$

Hence we may write

$$\psi = \frac{1}{c(\tau - z)} f(\Phi) + \frac{a}{c}\sqrt{\frac{2\Phi}{\tau - z}} \int_0^1 \mathrm{d}s\, f(\Phi(1 - s^2))\, I_1(a\sqrt{2(\tau - z)\Phi} \cdot s) \qquad \tau - r > 0. \tag{10}$$

In the particular case of $f = 1$ we obtain the simple expression

$$\psi = h(\tau - r)\frac{1}{c(\tau - z)} I_0\,(a\sqrt{\tau^2 - r^2}).$$

3. Applications of the obtained solution

Here we discuss how to use examples of the application of the results obtained in the previous section, for the description of transient and steady-state scalar and electromagnetic waves.

3.1. Transient and steady-state waves

Let us construct the solution of the telegraph equation supposing that $a > 0$ and that $f(\tau) = \exp(\mathrm{i}k\tau)$, where $k > 0$ is constant. Then, from (10), one obtains

$$\psi = \frac{1}{c(\tau - z)} \left[J_0(æ) + \mathrm{i}w \int_0^1 \mathrm{d}s\, s\, \mathrm{e}^{\frac{\mathrm{i}}{2}w(1-s^2)} J_0(æs) \right] \qquad \tau - r > 0 \tag{11}$$

where $w = 2k\Phi$, and $æ = \mathrm{i}a\sqrt{2(\tau - z)\Phi}$.

We express (11) in terms of Lommel's functions of two variables

$$U_n(w, æ) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{æ}\right)^{n+2m} J_{n+2m}(æ)$$

$$V_n(w, æ) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{æ}{w}\right)^{n+2m} J_{-(n+2m)}(æ)$$

by using the following relation (see 16.53 in [8])

$$\frac{w^n}{æ^{n-1}} \int_0^1 \mathrm{d}s\, s^n J_{n-1}(æs) \exp\left(\pm\frac{\mathrm{i}}{2}w(1 - s^2)\right) = U_n(w, æ) \pm \mathrm{i}U_{n+1}(w, æ). \tag{12}$$

In the case of $|w/æ| < 1$, from (11), we obtain

$$\psi = \frac{1}{c(\tau - z)} (U_0(w, æ) + \mathrm{i}U_1(w, æ)). \tag{13}$$

In the opposite case where $|w/æ| > 1$ we use the relation [8]

$$U_0(w, æ) + \mathrm{i}U_1(w, æ) = \mathrm{i}V_1(w, æ) + V_2(w, æ) + \exp\left(\frac{\mathrm{i}}{2}\left(w + \frac{æ^2}{w}\right)\right)$$

so that

$$\psi = \frac{1}{c(\tau - z)}(\mathrm{i}V_1(w, æ) + V_2(w, æ)) + \frac{1}{c(\tau - z)} \exp\left(\frac{\mathrm{i}}{2}\left(w + \frac{æ^2}{w}\right)\right) = \psi^{(1)} + \psi^{(2)}. \quad (14)$$

This expression describes both the transient and the steady-state processes. The latter term $\psi^{(2)}$ is the steady-state solution of the telegraph equation

$$\psi^{(2)} = \frac{1}{c(\tau - z)} \exp\left[\frac{\mathrm{i}}{2}\left(2k\Phi - \frac{a^2}{k}(\tau - z)\right)\right] \quad (15)$$

which, as it follows from the condition $|w/æ| > 1$, exists only inside the oblate ellipsoid of revolution around the z-axis

$$\frac{\rho^2}{(\tau/\sqrt{1+\alpha^2})^2} + \frac{(z - \alpha^2\tau/(1+\alpha^2))^2}{(\tau/(1+\alpha^2))^2} = 1 \qquad \alpha^2 = a^2/k^2 \quad (16)$$

Note that we obtain the general description of steady-state waves in dispersive media, including those akin to Brittingham's localized waves in free space, from expressions (13) and (14) replacing ik by a complex constant γ in (15) (see expression (26) in [5]). These waves exist inside the expanding surface (16) where the parameter $\alpha^2 = a^2/|\gamma|^2$.

3.2. *Scalar waves in lossy media*

Let us construct a solution where the initial-value problem for the telegraph equation with the dispersion term $2a\partial_\tau\psi$, where $a > 0$, that describes the formation of scalar waves in lossy media

$$\left(\partial_\tau^2 + 2a\partial_\tau - (1/\rho)\partial_\rho(\rho\partial\rho) - \partial_z^2\right)\psi = (4\pi/c)j \qquad \psi = j \equiv 0 \quad \tau < 0. \quad (17)$$

Putting

$$\psi = \exp(-a\tau)u(\rho, z, \tau)$$

we obtain

$$\left(\partial_\tau^2 - \partial_z^2 - (1/\rho)\partial_\rho(\rho\partial\rho) - a^2\right)u = (4\pi/c)\exp(a\tau)j \qquad u \equiv 0 \quad \tau < 0 \quad (18)$$

which enables us to write the solution of problem (17) by using expression (9) in the form

$$\psi = h(\tau - r)\frac{1}{c(\tau - z)}\,\mathrm{e}^{-a(\tau-\Phi)}f(\Phi) + \frac{a}{c}\sqrt{\frac{2\Phi}{\tau - z}}\,\mathrm{e}^{-a\tau}\int_0^\Phi \mathrm{d}\tau'\,\mathrm{e}^{a\tau'}f(\tau')\frac{1}{\sqrt{\Phi - \tau'}} \times I_1(a\sqrt{2(\tau - z)}\sqrt{\Phi - \tau'}). \quad (19)$$

Hence in the particular case of $f(\tau) = \exp(\mathrm{i}k\tau)$, by using expression (13) and (14), for $|w/æ| < 1$, we obtain

$$\psi = \frac{1}{c(\tau - z)}\,\mathrm{e}^{-a\tau}(U_0(w, æ) + \mathrm{i}U_1(w, æ)) \quad (20)$$

while for $|w/æ| > 1$ one has

$$\psi = \frac{1}{c(\tau - z)}\,\mathrm{e}^{-a\tau}(\mathrm{i}V_1(w, æ) + V_2(w, æ)) + \frac{1}{c(\tau - z)}\,\mathrm{e}^{-a\tau + \frac{\mathrm{i}}{2}\left(w + \frac{æ^2}{w}\right)}. \quad (21)$$

Here $æ = \mathrm{i}a\sqrt{2(\tau - z)\Phi}$, $w = 2(k - \mathrm{i}a)\Phi$, $|w/æ| = (1/\alpha)\sqrt{\tau^2 - r^2}/(\tau - z)$ and the parameter of the ellipsoid is $\alpha = a/\sqrt{k^2 + a^2}$.

The steady-state waves in lossy media are described by the latter term in (21)

$$\psi_l^{(2)} = \frac{1}{c(\tau - z)} \exp\left[-a\tau + \frac{\mathrm{i}}{2}\left(2(k - \mathrm{i}a)\Phi - \frac{a^2}{(k - \mathrm{i}a)}(\tau - z)\right)\right]. \quad (22)$$

Thus, in the case where $k = 0$, we get a localized wave inside the ellipsoid (16) with the parameter $\alpha = 1$

$$\psi_l^{(2)} = \frac{1}{c(\tau - z)} \exp\left[-\frac{1}{2} a \frac{\rho^2}{\tau - z}\right] \tag{23}$$

which differs from the axisymmetric wave of Brittingham's type. It is easy to verify that expression (23) satisfies the parabolic equation

$$\left((1/\rho)\partial_\rho(\rho\partial\rho) - 2a\partial_{\xi_1}\right)\psi_l^{(2)} = 0 \qquad \xi_1 = \tau - z \tag{24}$$

excluding the point ($\rho = 0, \xi_1 = 0$). One can treat the limit to this solution for $\tau \to \infty$ as a steady-state wave with the singularity $\delta(\rho)/\rho$ on the plane front $\xi_1 = 0$. Note that expression (23) has been obtained by Bateman as a specific solution of equation (24) [9].

3.3. Electromagnetic waves in conducting media

The results obtained permit us, in principle, to derive the electromagnetic vectors $\vec{E}$ and $\vec{B}$. We apply solutions of the inhomogeneous telegraph equation (17) to the description of TM electromagnetic waves in a conductor. Treating j as the z-component of the electric current density vector, we use the expressions [9]

$$E_\rho = \partial^2_{\rho z}\Pi \qquad E_z = \left(-\partial^2_\tau - 2a\partial_\tau + \partial^2_z\right)\Pi \qquad B_\varphi = -\partial_\rho(\partial_\tau + 2a)\Pi \tag{25}$$

where $a = (2\pi\sigma/c)$, σ is the conductivity and the scalar function Π is defined by the relation $(\partial_\tau + 2a)\Pi = \psi$. Then we obtain immediately the non-zero component of the magnetic induction vector. To find the electric-field strength components, one has to integrate the above relation with respect to the time variable. So, one can get the non-zero components of vectors $\vec{E}$, $\vec{B}$ via the expressions

$$E_\rho = -\partial^2_{\rho\xi_1}\Pi \qquad E_z = -2a\partial_{\xi_1}\Pi \qquad B_\varphi = -\left(\partial^2_{\rho\xi_1} + 2a\partial_\rho\right)\Pi$$

where the function $\Pi(\rho, \xi_1)$ is a solution of the problem

$$\left(\partial_{\xi_1} + 2a\right)\Pi = \psi_l^{(2)}(\xi_1, \rho) \qquad \Pi(0, \rho) = 0.$$

4. Concluding remarks

The method applied for solving the problem of wave formation by a point source moving with the velocity of light can also be used in the case of sources travelling slower or faster than light. We obtain the general solution to problem (1) supposing that the source function j is given by expression (2), where $\beta = v/c > 1$, v denotes the velocity of the point source. Then, in the space–time domain $r < \tau$, one has

$$\psi = \psi_0 + \psi_a = \frac{1}{cR} f(\Phi_+) + \frac{a}{c}\int_0^{\Phi_+} \mathrm{d}\tau' F(\tau', \tau, \rho, z) \tag{26}$$

while inside the circular cone with a vertex at the point $\rho = 0, z = \beta\tau$ and $z = -\rho\sqrt{\beta^2 - 1} + \beta\tau$ and outside the sphere $\tau = r$ the solution takes the form

$$\psi = \psi_0 + \psi_a = \frac{1}{cR}(f(\Phi_+) + f(\Phi_-)) + \frac{a}{c}\int_{\Phi_-}^{\Phi_+} \mathrm{d}\tau' F(\tau', \tau, \rho, z). \tag{27}$$

Here

$$R = \sqrt{(z - \beta\tau)^2 - (\beta^2 - 1)\rho^2}, \qquad \Phi_\pm = (-(\tau - \beta z) \pm R)/(\beta^2 - 1),$$

$$F = f(\tau')\frac{1}{\sqrt{(\tau - \tau')^2 - \rho^2 - (z - \beta\tau')^2}} I_1(a\sqrt{(\tau - \tau')^2 - \rho^2 - (z - \beta\tau')^2}).$$

In the case where $\tau < r$ the observation sector is limited by the conical surface $z = \rho/(\beta^2 - 1)^{1/2}$. The first terms in expressions (26) and (27) describe waves in free space [10]. The above solution requires an individual investigation, which is not a trivial problem.

On the basis of expressions (9) and (26) we can easily interpret similar solutions of the telegraph equation. (i) Solution (26) describes waves produced by a point source moving with a velocity less than the velocity of the wavefront (the velocity of light) when $\beta \in (0, 1)$. Hence, one obtains, (ii) the scalar wavefunction (9) in the limited case $\beta \to 1-$. (iii) Supposing that β is equal to zero, one gets wavefunction for a point source at rest from (26). This permits us to obtain (iv) the Green function for the 3D telegraph equation. Results (iii) and (iv) are given in [12].

Notably, the method discussed above for obtaining solutions (9), (26) and (27) is easier than that employing Green's function.

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Appendix

The generalization of the solution algorithm of the initial-value problem (1) is based on the theorem [1]:
Let $R_1(z, \tau; z', \tau')$ and $R_2(z, \tau; z', \tau')$ be the Riemann functions of the equations

$$\left(\partial_z^2 - \partial_\tau^2 + p_1(z)\right)\psi = 0 \quad \text{and} \quad \left(\partial_z^2 - \partial_\tau^2 + p_2(\tau)\right)\psi = 0.$$

Then the Riemann function of the equation

$$\left(\partial_z^2 - \partial_\tau^2 + p_1(z) + p_2(\tau)\right)\psi = 0$$

may be written as

$$\tilde{R} = R_1(z, \tau; z', \tau') + \int_{\tau-\tau'}^{z-z'} \mathrm{d}\xi\, R_1(z, \xi; z', 0)\frac{\partial}{\partial\xi}R_2(\xi, \tau; 0, \tau').$$

The application of the above theorem essentially simplifies the construction of solutions of different initial-value problems akin to problem (1) (see, for example, [6, 11]). Here, we have $p_1 = a^2$ and $p_2 = -s^2$, hence $R_1 = I_0(a\sqrt{(\tau-\tau')^2 - (z-z')^2})$ and $R_2 = J_0(s\sqrt{(\tau-\tau')^2 - (z-z')^2})$.

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